

Properties of Logarithms

The following properties serve to expand or condense a logarithm or logarithmic expression so it can be worked with.

Properties of logarithms

$$\log_a mn = \log_a m + \log_a n$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

$$\log_a m^n = n \log_a m$$

Example

$$\log_4 3x = \log_4 3 + \log_4 x$$

$$\log_2 \frac{x+1}{5} = \log_2 (x+1) - \log_2 5$$

$$\log_3 (2x+1)^3 = 3 \log_3 (2x+1)$$

Properties of Natural Logarithms

$$\ln mn = \ln m + \ln n$$

$$\ln \frac{m}{n} = \ln m - \ln n$$

$$\ln m^n = n \ln m$$

Example

$$\ln (x+1)(x-5) = \ln (x+1) + \ln (x-5)$$

$$\ln \frac{x}{2} = \ln x - \ln 2$$

$$\ln 7^3 = 3 \ln 7$$

These properties are used backwards and forwards in order to expand or condense a logarithmic expression. Therefore, these skills are needed in order to solve any equation involving logarithms. Logarithms will also be dealt with in Calculus. If a student has a firm grasp on these three simple properties, it will help greatly in Calculus.

Expanding Logarithmic Expressions

Write each of the following as the sum or difference of logarithms. In other words, expand each logarithmic expression.

A) $\log_2 \frac{3x^3y^2}{z^5} = \log_2 (3x^3y^2) - \log_2 z^5$
 $= \log_2 3 + \log_2 x^3 + \log_2 y^2 - \log_2 z^5$
 $= \log_2 3 + 3\log_2 x + 2\log_2 y - 5\log_2 z$

B) $\log_3 (5\sqrt[3]{xy^2}) = \log_3 5 + \log_3 (xy^2)^{\frac{1}{3}}$
 $= \log_3 5 + \frac{1}{3} [\log_3 x + \log_3 y^2]$
 $= \log_3 5 + \frac{1}{3} \log_3 x + \frac{2}{3} \log_3 y$

C) $\log \sqrt[4]{(x+1)^3(x-2)^2}$

$$\log [(x+1)^3(x-2)^2]$$

$$\frac{1}{4} \log [(x+1)^3(x-2)^2]$$

$$\frac{1}{4} [\log (x+1)^3 + \log (x-2)^2]$$

$$\frac{1}{4} [3\log (x+1) + 2\log (x-2)]$$

$$\frac{3}{4} \log (x+1) + \frac{1}{2} \log (x-2)$$

D) $\log_5 \frac{6x^2}{11y^5z}$

$$\text{E) } \log_2 \frac{\sqrt[5]{3(x+2)^3}}{x-1}$$

$$\text{F) } \log_{12} \frac{x-7}{x+2}$$

$$\text{G) } \log_a 12x^3 \sqrt{y}$$

$$\text{H) } \log_3 \frac{\sqrt{5x^5 y^3}}{\sqrt[3]{z^2}}$$

$$\text{I) } \ln \frac{x^2 - 4}{x^3}$$

$$\begin{aligned} \text{J) } \ln 3x^5 y &= \ln 3 + \ln x^5 + \ln y \\ &= \ln 3 + 5 \ln x + \ln y \end{aligned}$$

$$\text{K) } \ln \sqrt{x^3(x+4)}$$

$$\text{L) } \ln \sqrt[3]{x^2(x-3)}$$

$$\begin{aligned} \text{M) } \ln \sqrt{\frac{x^3 y}{z^5}} &= \ln \left(\frac{x^3 y}{z^5} \right)^{\frac{1}{2}} = \frac{1}{2} \ln \left(\frac{x^3 y}{z^5} \right) \\ &= \frac{1}{2} [\ln(x^3 y) - \ln z^5] = \frac{1}{2} \ln x^3 + \frac{1}{2} \ln y - \frac{1}{2} \ln z^5 \\ &= \frac{3}{2} \ln x + \frac{1}{2} \ln y - \frac{5}{2} \ln z \end{aligned}$$

$$\text{N) } \ln \frac{x}{\sqrt{x-2}}$$

Condensing Logarithmic Expressions

Rewrite each of the following logarithmic expressions using a single logarithm. Condense each of the following to a single expression. Do not multiply out complex polynomials. Just leave something like $(x+5)^3$ alone.

A) $3\log_4 x - 5\log_4 y + 2\log_4 z$
 $= \log_4 x^3 - \log_4 y^5 + \log_4 z^2$
 $= \log_4 (x^3 y^5 z^2)$

B) $2\log x + \frac{1}{2}\log y$

C) $\frac{1}{3}\log 6 + \frac{1}{3}\log x + \frac{2}{3}\log y$

D) $\frac{3}{4}\log_3 16 - \frac{1}{3}\log_3 x^3 - 2\log_3 y$

E) $3\log_5 x + 2\log_5 y + \log_5 z + 2$
 $= \log_5 x^3 + \log_5 y^2 + \log_5 z + 2$
 $= \log_5 (x^3 y^2 z) + 2$

F) $\frac{1}{3}\log_2 x + \frac{2}{3}\log_2 y - 3$

G) $\log_3 (x+2) + \log_3 (x-2) - \log_3 (x+4)$

H) $\frac{2}{3}\log (x+1) + \frac{1}{3}\log (x-2) - \frac{1}{3}\log (x+5)$

I) $\frac{1}{3}[2\ln(x+3) + \ln x] - \ln(2x-1)$

J) $\ln(x+3) - \ln(2x+5) + 2\ln(x-1)$
 $= \ln \frac{(x+3)}{(2x+5)} + \ln(x-1)^2$
 $= \ln \left[\frac{(x+3)(x-1)^2}{2x+5} \right]$

Solving Logarithmic Equations

Here we will solve some logarithmic equations. There are a couple of ways to solve a logarithmic equation. One method is to get all logs to the left side of the equal sign and condense the problem so there is only one log. The next step would be to write the problem in exponential form and solve.

Another method involves one of the properties of simple logarithms. In particular, the property that states: "If $\log_a x = \log_a y$ then $x = y$." If the equation can be manipulated to resemble this property, take what is inside of each and set the two terms equal to each other. This follows the same principal as exponential equations where the bases are identical. If an exponential equation states $2^{3x+4} = 2^7$; it stands to reason, because of the equal sign, that $3x+4=7$. The problem would then be solved from this point. With logarithmic equations, the problem $\log_4 (8x-5) = \log_4 12$ can be written as $8x-5=12$.

Always check for extraneous roots!!!

Solve each of the following logarithmic equations. (Round any solutions with decimals to three decimal places)

A) $\log_4 x + 6 = 8$

B) $2\log_3 x - 4 = 1$

$$\begin{aligned} 2\log_3 x &= 5 \\ \log_3 x^2 &= 5 \\ x^2 &= 3^5 \\ x^2 &= 243 \\ x &= \sqrt{243} \end{aligned}$$

$$\begin{array}{r} 3 \\ 3 \\ 3 \\ \hline 81 \\ \hline 243 \end{array}$$

C) $5 + 2\log_4 x = 4$

D) $\log_3 x - \log_3 2 = 5$

E) $\log(2x-1) = 0$

F) $\log(4x+8) = 2$

$$\begin{aligned} 4x+8 &= 10^2 \\ 4x+8 &= 100 \\ \frac{4x}{4} &= \frac{92}{4} \\ x &= 23 \end{aligned}$$

G) $\log_3(x+5) + \log_3(x+3) = \log_3 35$

$$\begin{aligned} \log_3[(x+5)(x+3)] &= \log_3 35 \\ x^2 + 8x + 15 &= 35 \\ x^2 + 8x - 20 &= 0 \\ (x+10)(x-2) &= 0 \\ x &= \cancel{-10}, 2 \end{aligned}$$

H) $\log(x+3) + \log(x-4) = \log 30$

I) $\log_3 3x - 2 = 2$

J) $\log_5 x + \log_5 (x+4) = 4\log_5 2$

K) $\log_x 625 = 4$

L) $\log_2 (x-2) + \log_2 (x-6) = \log_2 13$

M) $2\log_3 x - \log_3 (x-2) = 2$

N) $\ln x - \ln 2 = 0$

$$\begin{aligned} \ln \frac{x}{2} &= 0 \\ \frac{x}{2} &= e^0 \\ \frac{x}{2} &= 1 \\ x &= 2 \end{aligned}$$

O) $\ln(3x+5) = 8$

$$\begin{aligned} 3x+5 &= e^8 \\ 3x &= e^{8-5} \\ x &= \frac{e^{8-5}}{3} \end{aligned}$$

P) $\ln(x-1) = 3.2$

Q) $\log_3 (4x-2) = \log_3 (x+6)$

R) $\ln(3x-2) = 4$

S) $\ln e^{x+7} = 10$

$$\begin{aligned} x+7 &= 10 \\ x &= 3 \end{aligned}$$

T) $\log_a (x^2 + 7) = \frac{2}{3} \log_a 64$

U) $\log_2 (x+3) + \log_2 (x-3) = 4$

Solving Exponential Equations

You will now be solving exponential equations. Logarithms allow you to solve equations with variables in the exponent. We have dealt with these types of problems before, however, in the previous examples the bases of the problem could be made to match making solving the problem as simple as setting the two exponents equal to each other and solving for the variable. When ever you come to a problem with the variable in the exponent, and you cannot get the bases to match, you will need to solve the problem using logarithms. You will be using all three properties of logarithms to simplify and eventually solve the problem. You can take either the log or natural log of both sides, it makes no difference unless e is in the problem. In that case, you must use natural logs to solve the equation.

Solve each of the following exponential equations. Round solutions to three decimal places.

A) $3^{2x} = 5$

B) $6^{4x} = 300$

$$4x = \log_6 300$$
$$x = \frac{\log_6 300}{4}$$

C) $12^{3x+1} = 7^2$

D) $5^{x+2} = 3^5$

E) $4^{5x+8} = 8^{x-1}$

$$(2^2)^{5x+8} = (2^3)^{x-1}$$
$$10x+16 = 3x-3$$
$$7x = -19$$
$$x = -\frac{19}{7}$$

F) $9^{3x+2} = 27^{x+8}$

$$(3^2)^{3x+2} = (3^3)^{x+8}$$
$$6x+4 = 3x+24$$
$$3x = 20$$
$$x = \frac{20}{3}$$

G) $12^{3x-2} = 8^{5x+1}$

H) $5^{6x-5} = 6^{2x+1}$

I) $4e^{3x} = 40$

$$e^{3x} = 10$$
$$3x = \ln 10$$
$$x = \frac{\ln 10}{3}$$

J) $1200e^x = 900$

K) $3e^{4x} = 12$

$$e^{4x} = 4$$
$$4x = \ln 4$$
$$x = \frac{\ln 4}{4}$$

L) $e^{2x} - 3e^x + 2 = 0$

M) $e^{2x} - 5e^x + 6 = 0$

N) $e^{2x} + 2e^x - 24 = 0$

O) $2e^{2x} + e^x - 10 = 0$

P) $5^{2x} - 7 \cdot 5^x + 10 = 0$

Q) $3^{2x} + 5 \cdot 3^x - 14 = 0$

R) $\log_8(\log_4(\log_6 x)) = \frac{1}{3}$

S) $\log_2(\log_4 x) = 1$

T) $\log_3(\log_{27} x) = -1$