

Exponential Functions

General Form: $y = ab^x$, where $a \neq 0$, $b \neq 1$ and $b > 0$.

- b is the base and is a constant
- Domain is set of all reals; range $y > 0$; y-intercept $(0, a)$
- x is the independent variable with domain of x is set of all reals
- Exponential Growth: $a > 0$ and $b > 1$
- Exponential Decay: $a > 0$ and $0 < b < 1$
- Line $y = 0$ is called an *asymptote*.
- b – If Growth it's the *growth factor*; If Decay it's the *decay factor*

Examples:

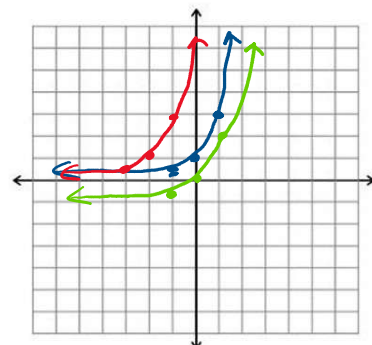
1. What is the graph of $y = 3^x$?

$$\frac{3^{-1}}{1}$$

x	y
-1	$3^{-1} = \frac{1}{3}$
0	1
1	3

$$y = 3^{x+2}$$

$$y = 3^x - 1$$



2. Identify $y = 0.7^x$ as an example of exponential growth or decay. What is the y-intercept?

$0 < b < 1 \therefore$ exponential decay

$$y = 0.7^0 = 1$$

$(0, 1)$ y-intercept

Another Model for Exponential Growth and Decay: $A(t) = a(1 + r)^t$

Another Model for Exponential Growth and Decay: $A(t) = a(1 + \frac{r}{n})^{nt}$

$A(t)$ is amount after t time periods

a is the initial amount

r is the rate of growth or decay

t is number of time periods

- Often used when doing financial/banking questions

Examples:

1. You buy a savings bond for \$25 that pays a yearly interest rate of 4.2%. What will the savings bond be worth after fifteen years?

$$a = 25$$

$$r = 0.042$$

$$t = 15$$

$$A(15) = 25(1 + 0.042)^{15}$$

$$= \$46.34$$

2. You open a savings account that pays 4.5% annual interest. If your initial investment is \$300 and you make no additional deposits or withdrawals, how many years will it take for the account to grow to at least \$500? (Calculator Required – Table)

$$500 \approx 300 \left(1 + \frac{0.045}{1}\right)^{1(t)}$$

$$t \approx 11.6 \text{ years}$$

Natural base exponential functions: Exponential functions base e . e is a number approximately 2.71828...

- Base e exponential functions are found in many applied real-world applications (biology, chemistry, banking, etc.)
- **Continuously Compounded Interest**
 - $A(t) = Pe^{rt}$

Graphing characteristics same as other shifts, etc.

Examples

1. How does the graph of $y = -\frac{1}{2} \cdot 4^x$ compare to the graph of the parent function?

flipped on x-axis, wider

2. How does the graph of $y = 3^{(x+1)}$ compare to the graph of the parent function?

Left 1 unit

3. Some insects reproduce exponentially. The chart shows the population of roaches in a colony at 36-day intervals. On what day will the colony reach 50,000,000 roaches? (Hint – ExpReg)

Day	Number of Roaches
1	50
37	1125
73	25,290
109	569,025
145	12,803,040

$$y = 45.86(1.09)^x$$

$$t = 160.76 \text{ days}$$

4. What is the value of $2e^6$? (Calculator)

$$806.86$$

5. You have \$1500 in a bank account that pays 4.5% annual interest compounded continuously. How much will you have in the account after 15 years? Round to the nearest dollar.

$$A(15) = 1500 e^{(.045)(15)} \approx \$2,946.05$$

$$\text{annual: } A(15) = 1500 \left(1 + \frac{.045}{1}\right)^{15} = 2902.92$$

$$\text{quarterly: } A(15) = 1500 \left(1 + \frac{.045}{4}\right)^{4(15)} = 2934.97$$

$$\text{daily: } A(15) = 1500 \left(1 + \frac{.045}{365}\right)^{365(15)} = 2945.93$$

$$1500 * \left(1 + (.045/365)\right)^{(365 * 15)}$$

Logarithm's and Properties

$$y = \log_a x \text{ if and only if } x = a^y$$

"y equals log base a of x"

Properties of Logarithms

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a a^x = x \rightarrow a^{\log_a x} = x$
- If $\log_a x = \log_a y$ Then $x = y$

EX: Simplify:

- $\log_9 9 = 1$
- $20^{\log_{20} 3} = 3$
- $\log_{\sqrt{3}} 1 = 0$
- Solve: $\log_5 (x^2 + 3) = \log_5 12$

$$\begin{aligned} x^2 + 3 &= 12 \\ -3 & \quad -3 \\ \hline x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \\ x &= \pm 3 \end{aligned}$$

EX: Evaluate:

- $f(x) = \log_6 x, x = 1$
 $\log_6 1 \stackrel{?}{=} 0$
- $f(x) = \log_5 x, x = \frac{1}{125}$
 $\log_5 \left(\frac{1}{125}\right) = \log_5 (125^{-1}) = \log_5 (5^{-3}) = -3$
- $f(x) = \log_7 x, x = 343$
 $\log_7 343 \stackrel{?}{=} \log_7 7^3 = 3$

Natural Logarithm: Log base e $\log_e x$

$$f(x) = \log_e x = \ln x$$

Properties of Natural Log

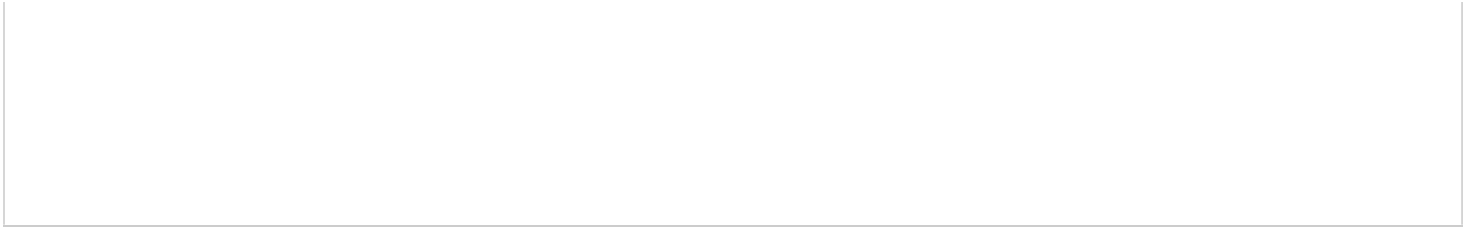
- $\ln 1 = 0$ $\log_e 1 = 0$
- $\ln e = 1$ $\log_e e = 1$
- $\ln e^x = x$ $\log_e e^x = x$
- If $\ln x = \ln y$ then $x = y$

EX: Simplify

- $\ln e^{\frac{1}{3}} = \frac{1}{3}$ $\log_e e^{\frac{1}{3}}$
- $5 \ln 1 = 0$ $5 \log_e 1$
- $\frac{3}{4} \ln e = \frac{3}{4}$ $\frac{3}{4} \log_e e$
- $e^{\ln 7} = 7$ $e^{\log_e 7}$

More Properties of Logarithms

- $\log_b mn = \log_b m + \log_b n$
- $\log_b \frac{m}{n} = \log_b m - \log_b n$
- $\log_b m^n = n \log_b m$
- $\log_b m = \frac{\log_c m}{\log_c b}$ (Change of Base)
- Log on your calculator is $\log_{10}$
- Examples:
 - What is each expression written as a single logarithm? If possible, simplify the single logarithm.
 - $\log_3 x - 2 \log_3 7 = \log_3 x - \log_3 7^2$
 $= \log_3 \left(\frac{x}{49} \right)$
 - $\log_8 48 + \log_8 \frac{4}{3} = \log_8 \left(\frac{48}{3} \cdot \frac{4}{3} \right)$
 $= \log_8 (64) = \log_8 8^2$
 $= \underline{2}$
 - What is each logarithm expanded? Simplify your answer, if possible.
 - $\log_5 \frac{125}{xy} = \log_5 (125) - \log_5 (xy)$
 $\log_5 5^3 - [\log_5 x + \log_5 y]$
 $\underline{3 - \log_5 x - \log_5 y}$



$$\begin{aligned} & \log x^2 y^2 z^{-1} \\ & \log x^2 + \log y^2 + \log z^{-1} \\ & 2\log x + 2\log y - 1\log z \end{aligned}$$

- What is the value of each expression? Use Change of Base and Calculator.

$$\log_9 111 = \frac{\log 111}{\log 9} = \frac{\ln 111}{\ln 9} = \frac{\log_{12} 111}{\log_{12} 9} = \frac{\log_7 111}{\log_7 9}$$

$$\begin{aligned} \log_{216} 36 &= \frac{\log_6 36}{\log_6 216} = \frac{\log_6 6^2}{\log_6 6^3} = \frac{2}{3} \\ &= \frac{\log_3 36}{\log_3 216} = \frac{\log_3 36}{\log_3 216} \end{aligned}$$

Solving Exponential and Logarithm Equations

- If possible, write both sides of the equation with same base, b .

- Example: What is the solution of $256^{2x} = 64$?

$$\begin{array}{r} 4 \checkmark \\ 4 \checkmark \\ \hline 16 \checkmark \\ 4 \checkmark \\ \hline 64 \checkmark \\ \hline 256 \end{array}$$

$$\begin{aligned} (4^4)^{2x} &= 4^3 \\ 4^{8x} &= 4^3 \\ \frac{8x}{8} &= \frac{3}{8} \\ x &= \frac{3}{8} \end{aligned}$$

- If not easily possible to write both sides of the equation with the same base, use logarithm properties to gain access to the exponent variable. Take log or ln of both sides.

- Example: What is the solution of $6^{4x} = 512$?

$$\begin{aligned} \log_6 6^{4x} &= \log_6 512 \\ \frac{4x}{4} &= \frac{\log_6 512}{4} \\ x &= \frac{\log_6 512}{4} = \frac{3.4816752651}{4} \approx 0.8704 \end{aligned}$$

- Example: What is the solution of $5^{2x} = 3500$? Use calculator method.

$$\begin{aligned} \log_5 5^{2x} &= \log_5 3500 \\ \frac{2x}{2} &= \frac{\log_5 3500}{2} \\ x &= \frac{\log_5 3500}{2} \approx 2.5352 \end{aligned}$$

- Example: What is the solution of $\log(5x + 2) = 2$?

$$10^{\log(5x+2)} = 10^2$$

$$\begin{array}{r} 5x + 2 = 100 \\ -2 \quad -2 \end{array}$$

$$\frac{5x}{5} = \frac{98}{5}$$

$$\boxed{x = \frac{98}{5}}$$

- Example: What is the solution of $\log 2x^2 - \log 5 = 1$?

$$\log_{10}\left(\frac{2x^2}{5}\right) = 1$$

$$10^{\log_{10}\left(\frac{2x^2}{5}\right)} = 10^1$$

$$\frac{2x^2}{5} = 10$$

$$x^2 = 25$$

$$\frac{2x^2}{2} = \frac{50}{2}$$

$$\boxed{x = \pm 5}$$

- Your phone has about 126,000,000 bytes of memory. Each month you plan to use 5% of the memory remaining. How many months will it take you to use $\frac{1}{4}$ of the memory?

Practice Worksheet: Evaluating Logarithms

Rewrite the equation in exponential form.

1] $\log_7 49 = 2$

$7^{\log_7 49} = 7^2 \rightarrow 49 = 7^2$

2] $\log_5 125 = 3$

3] $\log_4 \frac{1}{4} = -1$

4] $\log_2 16 = 4$

5] $\log_{16} 4 = \frac{1}{2}$

6] $\log_3 \frac{1}{9} = -2$

Rewrite the equation in logarithmic form.

7] $13^2 = 169$

$\log_{13} 13^2 = \log_{13} 169$
 $2 = \log_{13} 169$

8] $9^{3/2} = 27$

9] $4^{-3} = \frac{1}{64}$

10] $10^{-3} = 0.001$

11] $64^{\frac{1}{2}} = 8$

12] $9^{-2} = \frac{1}{81}$

13] $12^2 = 144$

14] $\left(\frac{1}{12}\right)^2 = \frac{1}{144}$

Evaluate the logarithm without using a calculator. Show work to support your answer.

15] $\log_9 81 = \log_9 9^2 = 2 \log_9 9 = \boxed{2}$	16] $\log_{27} 3 = x$ $27^{\log_{27} 3} = 27^x$ $3 = 27^x$ $3 = (3^3)^x$ $\frac{1}{3} = \frac{3x}{3}$ $x = \frac{1}{3}$	17] $\log_4 32 =$
18] $\log_8 1 = 0$	19] $\ln e^4 = \log_e e^4 = \boxed{4}$	20] $\log_8 4 = x$ $8^{\log_8 4} = 8^x$ $4 = 8^x$ $2^2 = 2^{3x}$ $2 = 3x$ $x = \frac{2}{3}$
21] $\log_3 \frac{1}{3} = \log_3 3^{-1} = -1$	22] $\log 1000 = \log_{10} 1000 = \log_{10} 10^3 = \boxed{3}$	23] $\log_{\frac{1}{2}} 128 =$ $\frac{\log_2 128}{\log_2 \frac{1}{2}} = \frac{\log_2 2^7}{\log_2 2^{-1}} = \frac{7}{-1} = \boxed{-7}$
24] $\log_4 2 = \frac{\log_2 2}{\log_2 4} = \frac{1}{\log_2 2^2} = \frac{1}{2} = \boxed{\frac{1}{2}}$	25] $\log_{25} 125 = \frac{\log_5 125}{\log_5 25} = \frac{\log_5 5^3}{\log_5 5^2} = \frac{3}{2}$ $25^{\log_{25} 125} = 25^x$ $125 = 25^x$ $5^3 = (5^2)^x$ $3 = \frac{2x}{2}$ $x = \frac{3}{2}$	26] $\log_3 \frac{1}{243} = \log_3 3^{-5} = \boxed{-5}$
27] $\log_4 64 = \log_4 4^3 = \boxed{3}$	28] $\log_{64} 4 = \frac{\log_4 4}{\log_4 64} = \frac{1}{\log_4 4^3} = \frac{1}{3} = \boxed{\frac{1}{3}}$	29] $\log_6 \frac{1}{216} = \log_6 6^{-3} = \boxed{-3}$

$\frac{3}{2}$ ✓
 $\frac{2}{3}$ ✓
 $\frac{3}{2}$ ✓
 $\frac{2}{3}$ ✓

Simplifying Logarithms

Evaluate each of the following logarithms without the use of a calculator.

A) $\log_3 81 =$ B) $\log_4 \frac{1}{2} = \frac{\log_2 2^{-1}}{\log_2 2^2} = -\frac{1}{2}$ C) $\log_{12} 144 = \frac{\log_2 12^2}{\log_2 12^1} = 2$ D) $\log_6 \frac{1}{36} =$

$\frac{-1(\log_2 2)}{2(\log_2 2)}$

E) $\log_{\frac{2}{3}} \frac{9}{4} =$ F) $\log_{0.25} 4 =$ G) $\log_3 -3 =$ H) $\log_8 4 =$

I) $\log_{81} \frac{1}{27} = \frac{\log_3 3^{-3}}{\log_3 3^4} = -\frac{3}{4}$ J) $\log_{\frac{1}{16}} 32 =$ K) $\log_4 0 =$ L) $\log_{10} 1 = 0$

M) $\log_4 \frac{1}{8} =$ N) $\log_{27} \frac{1}{3} =$ O) $\log_9 3 =$ P) $\log_6 6^{3x} = 3x$

Q) $\log_{36} \frac{1}{6} =$ R) $\log_{128} 2 = \frac{\log_2 2^1}{\log_2 2^7} = \frac{1}{7}$ S) $\log_{\frac{1}{4}} 16 =$ T) $\log_z z^{2x} =$

U) $\ln e^{12} = 12$ V) $3^{\log_3 5} = 5$ W) $\ln 1 =$ X) $e^{\ln 4x} = 4x$
 $e^{\log_e(4x)}$
m,k9

Y) $\log_2 16\sqrt{2} = \log_2 16 + \log_2 \sqrt{2} = \log_2 2^4 + \log_2 2^{\frac{1}{2}} = 4 + \frac{1}{2} = \frac{9}{2}$ Z) $\log_3 \sqrt[5]{9} = \left[\frac{1}{5}(\log_3 3^2)\right] = \left(\frac{1}{5}\right)2 = \frac{2}{5}$ a) $\log_3 9\sqrt[3]{3} = \log_3 3^2 + \log_3 3^{\frac{1}{3}} = 2 + \frac{1}{3} = \frac{7}{3}$

b) $\log_5 \frac{1}{\sqrt[3]{25}} =$ d) $e^{\ln 5x^2} = 5x^2$
 $e^{\log_e(5x^2)}$

Practice Problems

Solve the following equations:

Remember that the arguments of all logarithms must be greater than 0. Also exponentials in the form of a^x will be greater than 0. Be sure to check all your answers in the original equation.

1. $3^{x-1} = 81$

$3^{x-1} = 3^4$
 $x-1=4$
 $+1 +1$
 $x=5$

2. $8^x = 4$

Same base
 $(2^3)^x = 2^2$
 $3x=2$
 $x=\frac{2}{3}$

3. $e^x = 5$

$\log_e e^x = \log_e 5$
 $x = \log_e 5 = \ln 5 \approx 1.61$

4. $-14 + 3e^x = 11$

$3e^x = 25$
 $e^x = \frac{25}{3}$
 $x = \ln \frac{25}{3}$

5. $-6 + \ln 3x = 0$

$\ln 3x = 6$
 $3x = e^6$
 $x = \frac{e^6}{3}$

6. $\log_{10}(3x+1) = 2$

$3x+1 = 100$
 $3x = 99$
 $x = 33$

7. $\ln x - \ln 3 = 4$

$\ln \frac{x}{3} = 4$
 $\frac{x}{3} = e^4$
 $x = 3e^4$

8. $2 \ln 3x = 4$

$\ln 3x = 2$
 $3x = e^2$
 $x = \frac{e^2}{3}$

9. $5^{x+2} = 4$

10. $\ln(x+2)^2 = 6$

$2 \ln(x+2) = 6$
 $\ln(x+2) = 3$
 $x+2 = e^3$
 $x = -2 + e^3$

11. $4^{-3x} = 0.25$

12. $2e^{2x} - 5e^x - 3 = 0$

13. $\log_7 3 + \log_7 x = \log_7 32$

14. $2 \log_6 4x = 0$

15. $\log_2 x + \log_2(x-3) = 2$

$\log_2 [x(x-3)] = 2$
 $x(x-3) = 4$
 $x^2 - 3x - 4 = 0$
 $(x-4)(x+1) = 0$
 $x = 4$ or $x = -1$
 $x = 4$ is the only solution.

16. $\log_2(x+5) - \log_2(x-2) = 3$

17. $4 \ln(2x+3) = 11$

$\ln(2x+3) = \frac{11}{4}$
 $2x+3 = e^{\frac{11}{4}}$
 $2x = -3 + e^{\frac{11}{4}}$
 $x = \frac{-3 + e^{\frac{11}{4}}}{2}$

18. $\log x - \log 6 = 2 \log 4$

19. $2^x = 64$

20. $5^x = 25$

21. $4^{x-3} = \frac{1}{16}$

$4^{x-3} = 4^{-2}$
 $x-3 = -2$
 $x = 1$

22. $3^{x-2} = 81$

23. $\log_3 x = 5$

24. $\log_4 x = 3$

$x = 4^3 = 64$

25. $\log_2 2x = \log_2 100$

$2x = 100$
 $x = 50$

26. $\ln(x+4) = \ln 7$

$x+4 = 7$
 $x = 3$

27. $\log_3(2x+1) = 2$

$2x+1 = 9$
 $2x = 8$
 $x = 4$

28. $\log_5(x-10) = 2$

29. $3^x = 500$

30. $8^x = 1000$

31. $\ln x = 7.25$

32. $\ln x = -0.5$

33. $2e^{0.5x} = 45$

34. $100e^{-0.6x} = 20$

35. $12(1-4^x) = 18$

36. $25(1-e^t) = 12$

37. $\log 2x = 1.5$

38. $\log_2 2x = -0.65$

39. $\frac{1}{3} \log_2 x + 5 = 7$

$\frac{1}{3} \log_2 x = 2$
 $\log_2 x = 6$
 $x = 2^6 = 64$

40. $4 \log_5(x+1) = 4.8$

41. $\log_2 x + \log_2 3 = 3$

42. $2 \log_4 x - \log_4(x-1) = 1$

$$19. 2^x = 64$$

$$20. 5^x = 25$$

$$21. 4^{x-3} = \frac{1}{16}$$

$$4^{x-3} = 4^{-2}$$

$$x - 3 = -2$$

$$x = 1$$

$$40. 4 \log_5(x + 1) = 4.8$$

$$41. \log_2 x + \log_2 3 = 3$$

$$42. 2 \log_4 x - \log_4(x - 1) = 1$$

Write the following equalities in exponential form.

$$(1) \log_3 81 = 4 \quad (2) \log_7 7 = 1 \quad (3) \log_{\frac{1}{2}} \frac{1}{8} = 3 \quad (4) \log_3 1 = 0$$

$$(5) \log_4 \frac{1}{64} = -3 \quad (6) \log_6 \frac{1}{36} = -2 \quad (7) \log_x y = z \quad (8) \log_m n = \frac{1}{2}$$

Write the following equalities in logarithmic form.

$$(1) 8^2 = 64 \quad (2) 10^3 = 10000 \quad (3) 4^{-2} = \frac{1}{16} \quad (4) 3^{-4} = \frac{1}{81}$$

$$(5) \left(\frac{1}{2}\right)^{-5} = 32 \quad (6) \left(\frac{1}{3}\right)^{-3} = 27 \quad (7) x^{2z} = y \quad (8) \sqrt{x} = y$$

Solve the following logarithmic equations.

$$(1) \ln x = -3 \quad (2) \log(3x - 2) = 2$$

$$(3) 2 \log x = \log 2 + \log(3x - 4) \quad (4) \log x + \log(x - 1) = \log(4x)$$

$$(5) \log_3(x + 25) - \log_3(x - 1) = 3 \quad (6) \log_9(x - 5) + \log_9(x + 3) = 1$$

$$(7) \log x + \log(x - 3) = 1 \quad (8) \log_2(x - 2) + \log_2(x + 1) = 2$$

