

Rational Functions

Format: $y = \frac{P(x)}{Q(x)}$

- Domain: Set of all reals *except* those which make the denominator zero!!!!
- If there is no value(s) for which the denominator is zero, then the function is said to be continuous
- $y = \frac{x}{x^2+8}$
- If there is a value(s) where the denominator is zero, there is *discontinuity*.
 - *Removable Discontinuity*: Could you make the function continuous by redefining discontinuity point(s).

$$y = \frac{(x+3)(\cancel{x+2})}{(\cancel{x+2})}$$

Discontinuity at
 $x = -2$
Removable

What if you made $f(-2) = 1$? IE, do the y values get infinitely close to a value on both sides of the discontinuity point?

--Occur when you factor both the numerator and the denominator, and they share a factor (the discontinuity caused by this factor is removable)

- *Non-Removable Discontinuity*: If there is no value that you could replace in for the discontinuity point to make continuous.

Same as Vertical Asymptote

$$y = \frac{x+4}{x-2}$$

$x = 2$ non-removable discontinuity

- Asymptotes:
 - *Vertical*: Occur when denominator of rational function is zero. Vertical asymptotes occur at the line $x = \underline{\hspace{1cm}}$
 - *Horizontal*:
 - If power of numerator greater than power of denominator: No horizontal asymptote.
 - If power of denominator greater than power of numerator: Horizontal asymptote at line $y = \underline{0}$
 - If the powers are the same, horizontal asymptote at line $y = \frac{a}{b}$ where a and b are the leading coefficients of the numerator and denominator.

Examples:

1. What are the domain and points of discontinuity of $y = \frac{x^2+4x+4}{x+2}$? Are the points of discontinuity removable or non-removable? What are the x and y intercepts?

$$y = \frac{(x+2)(\cancel{x+2})}{(\cancel{x+2})}$$

$$D: \{ \mathbb{R} \mid x \neq -2 \}$$

x-intercepts: $(-2, 0)$ when Top is zero
But $x = -2$ not in the domain, so NO x-intercepts

y-intercepts: $(0, 2)$

Discontinuity at $x = -2$ removable

2. What are the vertical asymptotes of the graph $y = \frac{x-3}{(x^2-3x+2)(x^2-7x+12)}$?

$$y = \frac{(\cancel{x-3})}{(x-2)(x-1)(x-4)(\cancel{x-3})}$$

Vertical Asymptotes: $x = 2, 1, 4$
(same as non-removable discontinuities)

3. What is the horizontal asymptote of the graph of $y = \frac{x^2+1}{-3x+6}$? Graph the function.

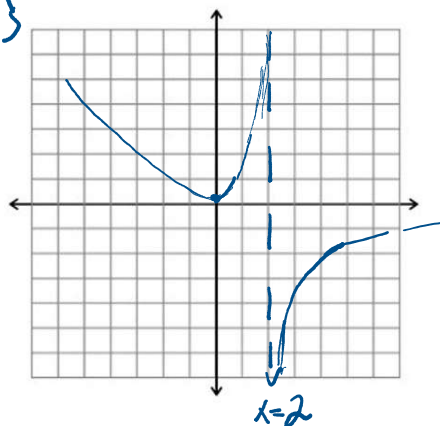
None! $D: \{R | x \neq 2\}$

Vertical? $-3x+6=0$
when $x=2$

x-intercepts: ~~$(0, 0)$~~

$$0 = \frac{x^2+1}{-3x+6}$$

y-intercept: $(0, \frac{1}{6})$



Rational Expressions & Equations

Rational Expression: Quotient of two polynomials.

- **Simplest Form:** When both numerator and denominator are polynomials that no longer share any common divisors/factors.
- **Domain:** Comes from non-simplified form
- **Adding/Subtracting:** Find LCM of denominator.
 - Factor expressions completely.
 - LCM of the denominator (LCD) is product of all prime factors (unique) to their highest power

Examples:

1. What is $\frac{9x^2+6x}{36x+24}$ in simplest form? State any restrictions on the variable.

$$\frac{3x(3x+2)}{12(3x+2)} = \frac{3x}{412} = \boxed{\frac{x}{4}} \quad x \neq -\frac{2}{3}$$

Restriction: $\frac{0}{0}$ $12(3x+2) \neq 0$ $3x = -2$ $x \neq -\frac{2}{3}$ (hole) (removable)

2. What is the product $\frac{x^2-3x+2}{x+2} \cdot \frac{x^2-36}{x^2+5x-6}$ in simplest form. State any restrictions on the variable.

$$\frac{(x-2)(x-1)}{(x+2)} \cdot \frac{(x+6)(x-6)}{(x-1)(x+6)} = \boxed{\frac{(x-2)(x-6)}{(x+2)}}$$

Restrictions: $x = -2, 1, -6$
 $x = 1, -6$ are removable discontinuities
 $x = -2$ non-removable (vertical asymptote)

3. What is the quotient $\frac{6x-3x^2}{36-x^2} \div \frac{x^3-x^2-2x}{x^2-5x-6}$ in simplest form? State any restrictions on the variable.

$$\frac{3x(2-x)}{-1(x^2-36)} \div \frac{x(x^2-x-2)}{(x+1)(x-6)} = \frac{3x(2-x)}{-1(x-6)(x+6)} \cdot \frac{(x+1)(x-6)}{x(x+1)(x-2)} = \boxed{\frac{3}{x+6}}$$

Restrictions: $x \neq 6, -6, -1$

4. What is the LCM of $x^2 + 4x - 12$ and $x^2 - 6x + 8$?

$$(x+6)(x-2) \quad (x-4)(x-2)$$

$$\text{LCD: } (x+6)(x-2)(x-4)$$

5. What is the sum $\frac{4}{x^2+3x} + \frac{x-2}{x^2+6x+9}$ in simplest form? State any restrictions on the variable.

$$\frac{4}{x(x+3)} + \frac{(x-2)}{(x+3)(x+3)} = \frac{4(x+3) + (x-2)x}{x(x+3)^2}$$

$$= \frac{4x+12+x^2-2x}{x(x+3)^2} = \frac{x^2+2x+12}{x(x+3)^2}$$

Restrictions: $x \neq 0, -3$
 $x = 0$; $x+3 = 0$ $x = -3$

$$\frac{x(x+3)}{x(x+3)^2} + \frac{(x-2)}{(x+3)^2} = \frac{4(x+3) + (x-2)x}{x(x+3)^2}$$

$$= \frac{4x+12+x^2-2x}{x(x+3)^2} = \frac{x^2+2x+12}{x(x+3)^2}$$

Restrictions: $x \neq 0, -3$
 $x=0$; $x+3=0$ $x=-3$

6. What is the difference $\frac{x+1}{x^2+2x-8} - \frac{x}{4x-8}$ in simplest form? State any restrictions on the variable.

$$\frac{(x+1)}{(x-2)(x+4)} - \frac{x}{4(x-2)} = \frac{(x+1)(4) - x(x+4)}{4(x-2)(x+4)}$$

$$\frac{4x+4-x^2-4x}{4(x-2)(x+4)} = \frac{-x^2+4}{4(x-2)(x+4)}$$

Restrictions: $x \neq 2, -4$

7. What is a simpler form of $\frac{3x - \frac{1}{y}}{\frac{y^2}{x} + x}$?

$$\frac{(3xy-1)}{y} \div \frac{y^2+x^2}{x} = \frac{(3xy-1)}{y} \cdot \frac{x}{y^2+x^2} = \frac{x(3xy-1)}{y(y^2+x^2)}$$

$\left[\frac{3xy-1}{y} \div \frac{y^2+x^2}{x} \right]$

Solving Rational Equations:

- Determine LCD.
- Multiple ALL terms by LCD.
- Simplify (you shouldn't have any denominators left).
- Solve.
- Check for extraneous solutions (often occur if one of your solutions is a value which your original equation is undefined).

Examples:

1. What are the solutions of the rational equation?

a. $\frac{4}{x+2} = \frac{5}{2x+3}$ LCD: $(x+2)(2x+3)$

$$\frac{(x+2)(4)}{(x+2)} = \frac{(x+2)(5)}{(2x+3)}$$

$$\begin{aligned} 4(2x+3) &= 5(x+2) \\ 8x+12 &= 5x+10 \\ -5x &= -2 \\ 3x &= -2 \\ x &= -\frac{2}{3} \end{aligned}$$

b. $\frac{1}{x^2-5x} + \frac{x-7}{x} = \frac{4}{x^2-5x}$ LCD: $x(x-5)$

$$x(x-5) \left[\frac{1}{x(x-5)} \right] + x(x-5) \left[\frac{x-7}{x} \right] = x(x-5) \left[\frac{4}{x(x-5)} \right]$$

$$\cancel{x(x-5)} \left[\frac{1}{\cancel{x(x-5)}} \right] + \cancel{x(x-5)} \left[\frac{x-7}{\cancel{x}} \right] = \cancel{x(x-5)} \left[\frac{4}{\cancel{x(x-5)}} \right]$$

$$1 + (x-5)(x-7) = 4$$

$$1 + x^2 - 7x - 5x + 35 = 4$$

$$\rightarrow x^2 - 12x + 32 = 0$$

$$(x-4)(x-8) = 0$$

$$x = 4, 8$$

c. $\frac{5}{x+1} + \frac{x}{x-1} = 5$